Jackson's paradox and its resolution by the four-dimensional geometric quantities

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In this paper it is shown that the real cause of Jackson's paradox is the use of three-dimensional (3D) quantities, e.g., $\mathbf{E}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{L}$, $\mathbf{T}$, their transformations and equations with them. The principle of relativity is naturally satisfied and there is no paradox when the physical reality is attributed to the 4D geometric quantities, e.g., to the 4D torque N (bivector) or, equivalently, to the 4D torques N_s and N_t (1-vectors), which together contain the same physical information as the bivector N.

I. INTRODUCTION

In a recent paper[1] in this Journal Jackson discovered an apparent paradox; there is a three-dimensional (3D) torque and so a time rate of change of 3D angular momentum in one inertial frame, but no 3D angular momentum and no 3D torque in another. Two inertial frames S (the laboratory frame) and S' (the moving frame) are considered (they are K and K' respectively in Jackson's notation). In S' a particle of charge q and mass m experiences only the radially directed electric force caused by a point charge Q fixed permanently at the origin. Consequently both $\mathbf{L}'$ and the torque $\mathbf{T}'$ (Jackson's $\mathbf{N}'$ is denoted as $\mathbf{T}'$) are zero in S', see Fig. 1(a) in Ref. 1. (The vectors in the 3D space will be designated in bold-face.) In S the charge Q is in uniform motion and it produces *both* an electric field $\mathbf{E}$ and a *magnetic field* $\mathbf{B}$. The existence of $\mathbf{B}$ in S is responsible for the existence of the 3D magnetic force $\mathbf{F} = q\mathbf{u} \times \mathbf{B}$ and this force provides a 3D torque $\mathbf{T}$ ($\mathbf{T} = \mathbf{x} \times \mathbf{F}$) on the charged particle, see Fig. 1(b) in Ref. 1. Consequently a nonvanishing angular momentum of the charged particle changes in time in S, $\mathbf{T} = d\mathbf{L}/dt$. Jackson[1] considers that there is no paradox and that such result is relativistically correct result, i.e., that the principle of relativity is not violated.

It is recently revealed[2] that the real cause of the paradox is - the use of 3D quantities, e.g., $\mathbf{E}$, $\mathbf{B}$, $\mathbf{F}$, $\mathbf{L}$, $\mathbf{T}$, their transformations and equations with them. In Ref. 2, instead of using 3D quantities, it is dealt from the outset with 4D geometric quantities and equations with them. In such treatment the paradox does not appear and the principle of relativity is naturally satisfied. In this paper we shall briefly repeat the consideration from Ref. 2. It is worth noting that exactly the same paradox appears in the Trouton-Noble experiment, see, e.g., Ref. 3 and references therein.

The calculations in Ref. 2 (also in Ref. 3) and in this paper are performed in the geometric algebra formalism, which is recently nicely presented in this

Journal by Hestenes.[4] Physical quantities will be represented by geometric 4D quantities, multivectors, that are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, they are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis. For simplicity and for easier understanding only the standard basis $\{\gamma_\mu; 0, 1, 2, 3\}$ of orthonormal 1-vectors, with timelike vector γ_0 in the forward light cone, will be used. For all mathematical details regarding the spacetime algebra reader can consult Hestenes' paper.[4]

II. DISCUSSION OF JACKSON'S DERIVATION AND RESULTS

First a remark about the figures in Ref. 1. Both figures would need to contain the time axes as well. Fig. 1(a) (Fig. 1(b), and Fig. 2) is the projection onto the hypersurface $t' = const.$ ($t = const.$); the distances are simultaneously determined in the S' (S) frame. The Lorentz transformations (LT) cannot transform the hypersurface $t' = const.$ into the hypersurface $t = const.$ and the distances that are simultaneously determined in the S' frame cannot be transformed by the LT into the distances simultaneously determined in the S frame. Such figures could be possible for the Galilean transformations, but for the LT they are meaningless.

In Sec. III Jackson[1] discusses "Lorentz transformations of the angular momentum between frames." He starts with the usual covariant definition of the angular momentum tensor $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, Eq. (8) in Ref. 1. Notice that the standard basis $\{\gamma_\mu\}$, i.e., Einstein's system of coordinates, is implicit in that definition. In Einstein's system of coordinates the standard, i.e., Einstein's synchronization[5] of distant clocks and Cartesian space coordinates x^i are used in the chosen inertial frame. In Ref. 1 the vector $\mathbf{L}$ is called the angular momentum and its components L_i are identified with the space-space components of $M^{\mu\nu}$. However, a physical interpretation is not given for another vector $\mathbf{L}_t$. Its components $L_{t,i}$ are identified with the three time-space components of $M^{\mu\nu}$ (we denote Jackson's K_i with $L_{t,i}$, $\mathbf{K}$ with $\mathbf{L}_t$). The same identification is supposed to hold both in S', the rest frame of the charges q and Q, $L'_i = (1/2)\varepsilon_{ikl}M'^{kl}$, $L'_{t,i} = M'^{0i}$, and in S, the laboratory frame; the same relations but with unprimed quantities. This then leads to the usual transformations of the components of $\mathbf{L}$ that are given by Eq. (11) in Ref. 1. In contrast to Ref. 1 we write both transformations, for L_i and for $L_{t,i}$; $L_1 = L'_1$, $L_2 = \gamma(L'_2 - \beta L'_{t,3})$, $L_3 = \gamma(L'_3 + \beta L'_{t,2})$, and $L_{t,1} = L'_{t,1}$, $L_{t,2} = \gamma(L'_{t,2} + \beta L'_3)$, $L_{t,3} = \gamma(L'_{t,3} - \beta L'_2)$. The characteristic feature of these transformations is that the components L_i in S are expressed by the mixture of components L'_i and $L'_{t,i}$ from S'. This causes that the components of the 3D angular momentum do not vanish in the laboratory frame S, even if they do in S'. In the case considered in Ref. 1 L_3 is different from zero due to contribution from $L'_{t,2}$.

This is in a complete analogy with the usual identification of the components of $\mathbf{B}$ and $\mathbf{E}$ with the space-space and the time-space components respectively of the electromagnetic field strength tensor $F^{\mu\nu}$. In S' it is $B'_i = (1/2c)\varepsilon_{ikl}F'^{lk}$,

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$E'_i = F'^{i0}$, and the same relations but with unprimed quantities hold in S, see Ref. 6 Sec. 11.9. Such procedure yields the usual transformations for the components of $\mathbf{B}$ and $\mathbf{E}$, see, e.g., Ref. 6 Eq. (11.148). The comparison of the identifications shows that the components L_i correspond to $-B_i$ and $L_{t,i}$ to $-E_i$. In all these relations the components of the 3D vectors $\mathbf{L}$, $\mathbf{L}_t$ and $\mathbf{B}$, $\mathbf{E}$ are written with lowered (generic) subscripts, since they are not the spatial components of the 4D quantities. This refers to the third-rank antisymmetric ε tensor too. The super- and subscripts are used only on the components of the 4D quantities.

In the same way we can determine the components T_i of the torque $\mathbf{T}$ ($\mathbf{T} = d\mathbf{L}/dt$) and the components $T_{t,i}$ of another torque $\mathbf{T}_t$ ($\mathbf{T}_t = d\mathbf{L}_t/dt$) by the identification in both frames with the space-space and the time-space components respectively of the torque four-tensor $N^{\mu\nu}$, $T_i = (1/2)\varepsilon_{ikl}N^{kl}$, $T_{t,i} = T^{0i}$, which gives the transformations

$$\begin{aligned} T_1 &= T'_1, \ T_2 = \gamma(T'_2 - \beta T'_{t,3}), \ T_3 = \gamma(T'_3 + \beta T'_{t,2}), \\ T_{t,1} &= T'_{t,1}, \ T_{t,2} = \gamma(T'_{t,2} + \beta T'_3), \ T_{t,3} = \gamma(T'_{t,3} - \beta T'_2). \end{aligned} \tag{1}$$

Again the transformed components T_i are expressed by the mixture of components T'_k and $T'_{t,k}$. Hence the components of "physical" torque $\mathbf{T}$ do not vanish in S, even if they do in S'. In Ref. 1 all T'_k are zero but the component T_3 in S is $\neq 0$ due to contribution from $T'_{t,2}$. Jackson,[1,6] as all others, considers that only the space-space parts, L_i and T_i, i.e. the 3D vectors $\mathbf{L}$ and $\mathbf{T}$, are physical quantities. Actually, it is almost generally accepted that the covariant quantities, e.g., $M^{\mu\nu}$, $N^{\mu\nu}$, $F^{\mu\nu}$, etc. are only auxiliary mathematical quantities from which "physical" 3D quantities, $\mathbf{L}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{B}$, etc., are deduced.

Several objections are raised in Refs. 2, 7-9 to such derivation of the usual transformations for the components of $\mathbf{L}$ and $\mathbf{L}_t$, $\mathbf{B}$ and $\mathbf{E}$ and $\mathbf{T}$ and $\mathbf{T}_t$. Some of them are quoted here.

(i) $M^{\mu\nu}$, $F^{\mu\nu}$ and $N^{\mu\nu}$ are only components (numbers) that are (implicitly) determined in the standard basis $\{\gamma_\mu\}$, i.e., in Einstein's system of coordinates. In another system of coordinates that is different than the Einstein system of coordinates, e.g., differing in the chosen synchronization, all above identifications are impossible and meaningless.

(ii) The above identifications and transformations refer only to components. The 3D vectors, e.g., $\mathbf{T}'$ and $\mathbf{T}'_t$ in S', and $\mathbf{T}$ and $\mathbf{T}_t$ in S, which are geometric quantities in the 3D space, are constructed multiplying six independent components of $N'^{\mu\nu}$ and $N^{\mu\nu}$ by the unit 3D vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ in S' and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in S. The components, e.g., T_i in S are determined by the usual transformations (1) from the components T'_i and $T'_{t,i}$ in S', but there is no transformation which transforms $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ from S' into $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in S. Hence it is not true that $\mathbf{T} = T_1\mathbf{i} + T_2\mathbf{j} + T_3\mathbf{k}$ is obtained by the LT from $\mathbf{T}' = T'_1\mathbf{i}' + T'_2\mathbf{j}' + T'_3\mathbf{k}'$. Consequently $\mathbf{T}$ and $\mathbf{T}'$ are not the same quantity for relatively moving inertial observers in S and S', $\mathbf{T} \neq \mathbf{T}'$. The transformations that do not refer to the same 4D quantity are not the LT, but we call them the "apparent" transformations (AT). According to that the transformations (1) and those for $\mathbf{L}$, Eq. (11)

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in Ref. 1, are not the LT, but they are the AT of the 3D **T** and **L**. In Ref. 7-9 it is explained in detail that the usual transformations of **B** and **E**, Ref. 6 Eqs. (11.148) and (11.149), are also the AT and not the LT (a fundamental achievement). In fact, as shown in Ref. 2, all transformations of the 3D vectors **p**, **F**, **E**, **B**, **L**, **T**, etc. are the AT and not the LT. This also means that equations with them, like Eq. (4) ($d\mathbf{p}/dt = \mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B}$) and Eq. (5) ($d\mathbf{L}/dt = \mathbf{T}$) in Ref. 1, are not relativistically correct equations. Thus, the fact that in Ref. 1 the same result, $\mathbf{T}' = \mathbf{0}$ but $\mathbf{T} \neq \mathbf{0}$, is obtained by two different methods of calculation does not mean that the calculations are relativistically correct and that the principle of relativity is satisfied.

III. DEFINITIONS OF 4D QUANTITIES

The relativistically correct treatment of the problem considered in Ref. 1 and the resolution of Jakson's paradox are given in Ref. 2, Secs. 4-4.3, using 4D torques N, N_s and N_t. Here we shall only briefly expose the consideration from Secs. 2, 4, 4.1 and 4.3, Ref. 2.

As shown in Ref. 10 the electromagnetic field F (bivector) is the primary quantity for the whole electromagnetism. $F(x)$ for a charge Q moving with constant velocity u_Q (1-vector) is

$$F(x) = kQ(x \wedge (u_Q/c))/\left|x \wedge (u_Q/c)\right|^3,\tag{2}$$

where $k = 1/4\pi\varepsilon_0$. For the charge Q at rest, $u_Q/c = \gamma_0$. Instead of the usual identification of E_i and B_i with components of $F^{\mu\nu}$ we construct, in a mathematically correct way, the 4D geometric quantities that represent the electric and magnetic fields by a decomposition of F. It can be decomposed into 1-vectors of the electric field E and the magnetic field B and a unit time-like 1-vector v/c as

$$F = (1/c)E \wedge v + (IB) \cdot v,$$
$$E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v),\tag{3}$$

see Ref. 8. In (3) I is the unit pseudoscalar. (I is defined algebraically without introducing any reference frame, as in Ref. 11, Sec. 1.2.) v is the velocity (1-vector) of a family of observers who measures E and B fields. Since F is antisymmetric it holds that $E \cdot v = B \cdot v = 0$, which yields that only three components of E and three components of B are independent quantities. Observe that E and B depend not only on F but on v as well.

From (3) and the known F, Eq. (2), we find the 1-vectors E and B for a charge Q moving with constant velocity u_Q

$$\begin{aligned} E &= (D/c^2)[(x \wedge u_Q) \cdot v] \\ B &= (-D/c^3)I(x \wedge u_Q \wedge v), \end{aligned}\tag{4}$$

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where $D = kQ/|x \wedge (u_Q/c)|^3$. Note that B in (4) can be expressed in terms of E as $B = (1/c^3)I(u_Q \wedge E \wedge v)$. When the world lines of the observer and the charge Q coincide, $u_Q = v$, then (4) yields that $B = 0$ and only an electric field (Coulomb field) remains. E and B from (4) are AQs; they are independent of the chosen reference frame and the chosen system of coordinates in it. They naturally generalize the expressions for the 3D **E** and **B**, Eqs. (3a) and (3b) in Ref. 1, to the 4D spacetime.

Furthermore, instead of the covariant form of the Lorentz force, Eq. (A2) in Ref. 1 (only components in the implicit $\{\gamma_\mu\}$ basis), we deal with the Lorentz force as an AQ, $K_L = (q/c)F \cdot u$. Using the decomposition of F into E and B, Eq. (3), this Lorentz force can be written as $K_L = (q/c)\left[(1/c)E \wedge v + (IB) \cdot v\right] \cdot u$, where u is the velocity (1-vector) of a charge q. Particularly, from the definition of the Lorentz force $K_L = (q/c)F \cdot u$ and the relation $E = (1/c)F \cdot v$ it follows that the Lorentz force ascribed by an observer comoving with a charge, $u = v$, is purely electric $K_L = qE$. Hence our equation of motion of a charge q is $m\,du/d\tau = K_L = (q/c)\left[(1/c)E \wedge v + (IB) \cdot v\right] \cdot u$, which replaces Eq. (4) from Ref. 1. In that equation τ is the proper time, u is the velocity 1-vector of a particle that is defined to be the tangent to its world line.

The 4D AQs, the angular momentum M and the torque N (bivectors) for the Lorentz force K_L and manifestly Lorentz invariant equation connecting M and N are defined as

$$M = x \wedge p, \quad N = x \wedge K_L; \quad N = dM/d\tau, \tag{5}$$

where x is the position 1-vector and p is the proper momentum (1-vector) $p = mu$.

When M and N are written as CBGQs in the $\{\gamma_\mu\}$ basis they become $M = (1/2)M^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, with $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, and $N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, with $N^{\mu\nu} = x^\mu K_L^\nu - x^\nu K_L^\mu$. The components, e.g., $N^{\mu\nu}$, are determined as $N^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot N) = (\gamma^\nu \wedge \gamma^\mu) \cdot N$. We see that the components $M^{\mu\nu}$ are identical to the covariant angular momentum four-tensor given by Eq. (A3) in Ref. 1. However M and N from (5) are 4D geometric quantities, the 4D AQs, whereas the components $M^{\mu\nu}$ and $N^{\mu\nu}$ that are used in the usual covariant approach, e.g., Eq. (A3) in Ref. 1, are coordinate quantities, the numbers obtained in the specific system of coordinates, i.e., in the $\{\gamma_\mu\}$ basis. In contrast to the usual covariant approach, M and N as 4D CBGQs are also 4D geometric quantities, which contain both components and a *basis*, here bivector basis $\gamma_\mu \wedge \gamma_\nu$. The essential difference between our geometric approach and the usual covariant picture is the presence of the basis. *The existence of a basis causes that every 4D CBGQ is invariant under the passive LT*; the components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. This means that a CBGQ represents *the same physical quantity* for relatively moving 4D observers. Hence it holds that, e.g.,

$$N = (1/2)N^{'\mu\nu}\gamma'_\mu \wedge \gamma'_\nu = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \tag{6}$$

where all primed quantities are the Lorentz transforms of the unprimed ones. When physical laws are written with such Lorentz invariant quantities, 4D AQs

or 4D CBGQs, then they automatically satisfy the principle of relativity. In the standard approach to special relativity[5] the principle of relativity is postulated outside the framework of a mathematical formulation of the theory. There,[5] it is also considered that the principle of relativity holds for the equations written with the 3D quantities.

Here, as in Ref. 2, we introduce new 4D torques as AQs, 1-vectors N_s and N_t. The same decomposition can be made for N as for F, (3). It is decomposed into two 1-vectors, the "space-space" torque N_s and the "time-space" torque N_t, and the unit time-like 1-vector v/c as

$$
\begin{aligned}
N &= (v/c) \cdot (IN_s) + (v/c) \wedge N_t \\
N_s &= I(N \wedge v/c), \ N_t = (v/c) \cdot N; \ N_s \cdot v = N_t \cdot v = 0.
\end{aligned}
\tag{7}
$$

Only three components of N_s and three components of N_t are independent since N is antisymmetric. Here again v is the velocity (1-vector) of a family of observers who measures N_s and N_t. Again, as for E and B, the torques N_s and N_t depend not only on the bivector N but on v as well. The relations (7) show that N_s and N_t taken *together* contain the same physical information as the bivector N.

When N_s and N_t are written as CBGQs in the $\{\gamma_\mu\}$ basis they are $N_s = N_s^\mu \gamma_\mu = (1/2c)\varepsilon^{\alpha\beta\mu\nu}N_{\alpha\beta}v_\mu\gamma_\nu$, $N_t = (1/c)N^{\mu\nu}v_\mu\gamma_\nu$. In the frame of "fiducial" observers (it will be called the γ_0-frame), in which the observers who measure 1-vectors E, B, K_L, N_s and N_t are at rest, the velocity v is $v = c\gamma_0$. In that frame and in the $\{\gamma_\mu\}$ basis $v^\mu = (c,0,0,0)$. Let us now take that the S frame is the γ_0-frame. Then in S, $N_s^0 = N_t^0 = 0$, and only the spatial components remain. N_s^i components are $N_s^1 = N^{23} = x^2K_L^3 - x^3K_L^2$, $N_s^2 = N^{31}$ and $N_s^3 = N^{12}$. Comparison with the identification $T_i = (1/2)\varepsilon_{ikl}N^{kl}$ shows that in the γ_0-frame the components of the 1-vector N_s correspond to components of "physical" 3D torque $\mathbf{T}$, $T_i = N_s^i$, and similarly $T_{t,i} = N_t^i$. Hence only for "fiducial" observers one can deal with *components* of *two* 3D torques $\mathbf{T}$ and $\mathbf{T}_t$; all *six* components are equally well physical. However, even in that frame the relativistically correct *geometric* quantities are not 3D vectors $\mathbf{T}$ and $\mathbf{T}_t$, but 1-vectors N_s and N_t. The whole discussion with the torque can be completely repeated for the angular momentum replacing N, N_s and N_t by M, M_s and M_t. In the γ_0-frame components of 1-vectors M_s and M_t correspond to components of $\mathbf{L}$ and $\mathbf{L}_t$ respectively in the usual 3D picture.

Of course, it holds that, e.g., N_s as a CBGQ is a Lorentz invariant quantity, $N_s = N_s^\mu \gamma_\mu = N_s'^\mu \gamma_\mu'$, where again all primed quantities are the Lorentz transforms of the unprimed ones. The components of N_s (N_t, M_s, M_t, E, B, ..) transform under the LT as the components of any 1-vector transform

$$
N_s'^0 = \gamma(N_s^0 - \beta N_s^1), \ N_s'^1 = \gamma(N_s^1 - \beta N_s^0), \ N_s'^{2,3} = N_s^{2,3}.
\tag{8}
$$

The LT of N_t^μ, M_s^μ, M_t^μ and of E^μ, B^μ are of the same form. In contrast to all mentioned AT for the components of the 3D vectors, e.g., Eq. (1) or Eq. (11) in Ref. 1 for L_i, the components N_s^μ transform again to $N_s'^\mu$; *there is no mixing of components.*

Furthermore we write N from (5) using the expression for K_L and Eq. (2) for F,

$$N = (Dq/c^2)(u \cdot x)(u_Q \wedge x). \qquad (9)$$

N_s and N_t are then determined from (7) and (9)

$$
\begin{aligned}
N_s &= (Dq/c^3)(u \cdot x)I(x \wedge v \wedge u_Q), \\
N_t &= (Dq/c^3)(u \cdot x)[(x \wedge u_Q) \cdot v].
\end{aligned}
\qquad (10)
$$

Comparison with (4) shows that N_s and N_t can be expressed in terms of B and E as

$$
\begin{aligned}
N_s &= q(u \cdot x)B, \\
N_t &= (q/c)(u \cdot x)E.
\end{aligned}
\qquad (11)
$$

As already said, in connection with (4), when $u_Q = v$ then $B = 0$ and $N_s = 0$ as well.

IV. RESOLUTION OF JACKSON'S PARADOX USING 4D TORQUES

The knowledge of N as an AQ, Eq. (9), enables us to find the expressions for N as CBGQs in S' and S. First let us write all AQs from (9) as CBGQs in S', the rest frame of the charge Q, in which $u_Q = c\gamma_0'$. Then $N = (Dq/c)(u \cdot x)(\gamma_0' \wedge x)$, and in the $\{\gamma_\mu'\}$ basis it is explicitly given as

$$
\begin{aligned}
N &= (1/2)N'^{\mu\nu}\gamma_\mu' \wedge \gamma_\nu' = N'^{01}(\gamma_0' \wedge \gamma_1') + N'^{02}(\gamma_0' \wedge \gamma_2'), \\
N'^{01} &= (Dq/c)(u'^\mu x_\mu')x'^1, \ \ N'^{02} = (Dq/c)(u'^\mu x_\mu')x'^2.
\end{aligned}
\qquad (12)
$$

The components x'^μ are $x'^\mu = (x'^0 = ct', x'^1, x'^2, 0)$ where $x'^1 = r'\cos\theta'$, $x'^2 = r'\sin\theta'$. In S', $u = u'^\mu \gamma_\mu'$, where $u'^\mu = dx'^\mu/d\tau = (u'^0, u'^1, u'^2, 0)$. The components $N'^{\mu\nu}$ that are different from zero are only N'^{01} and N'^{02}.

In order to find the torque N in S we now write all AQs from (9) as CBGQs in S and in the $\{\gamma_\mu\}$ basis. In S the charge Q is moving with velocity $u_Q = \gamma_Q c\gamma_0 + \gamma_Q \beta_Q c\gamma_1$, where $\beta_Q = |\mathbf{u}_Q|/c$ and $\gamma_Q = (1 - \beta_Q^2)^{-1/2}$. Then N is

$$N = (1/2)N^{\mu\nu}\gamma_\mu \wedge \gamma_\nu = N^{01}\gamma_0 \wedge \gamma_1 + N^{02}\gamma_0 \wedge \gamma_2 + N^{12}\gamma_1 \wedge \gamma_2, \qquad (13)$$

or explicitly it becomes

$$
\begin{aligned}
N &= (Dq/c)(u^\mu x_\mu)[\gamma_Q(x^1 - \beta_Q x^0)(\gamma_0 \wedge \gamma_1) \\
&\quad + \gamma_Q x^2(\gamma_0 \wedge \gamma_2) + \beta_Q \gamma_Q x^2(\gamma_1 \wedge \gamma_2)].
\end{aligned}
\qquad (14)
$$

Now the components $N^{\mu\nu}$ that are different from zero are not only the time-space components N^{01} and N^{02} but also the space-space component $N^{12} = \beta_Q N^{02}$.

We note that another way to find N as CBGQ in S, (14), is to make the LT of N as CBGQ in S', (12). Of course, according to (6), both CBGQs are equal, $N\,((12)) = N\,((14))$; they represent the same 4D quantity N from (9) in S' and S frames. The principle of relativity is naturally satisfied and there is not any paradox.

Instead of the torque N we can use N_s and N_t. However, as already said, N_s and N_t are not uniquely determined by N, but their explicit values depend also on v. This means that it is important to know which frame is chosen to be the γ_0-frame. Observe that the same conclusions also refer to the determination of E, B and M_s, M_t from F and M respectively. In this paper we shall only consider the case when S is the γ_0-frame. For another case readers can consult Sec. 4.2 in Ref. 2. N_s and N_t will be determined directly from (10) taking into account that $v = c\gamma_0$ and $u_Q = \gamma_Q c\gamma_0 + \gamma_Q \beta_Q c\gamma_1$. This yields that

$$N_s = N_s^\mu \gamma_\mu = N^{12}\gamma_3, \;\; N_t = N_t^1 \gamma_1 + N_t^2 \gamma_2 = N^{01}\gamma_1 + N^{02}\gamma_2, \qquad (15)$$

where N^{12}, N^{01} and N^{02} are from (14). Thus when S is the γ_0-frame *the "space-space" torque N_s is different from zero.*

The same N_s and N_t can be determined using (11) and (4). The charge Q moves in S, which yields that both E and *the magnetic field B are different from zero.* Then $E = E^\mu \gamma_\mu$, $E^0 = E^3 = 0$, $E^1 = D\gamma_Q(x^1 - \beta_Q x^0)$, $E^2 = D\gamma_Q x^2$, and the magnetic field is $B = B^\mu \gamma_\mu$, $B^0 = B^1 = B^2 = 0$, $B^3 = (D/c)\gamma_Q \beta_Q x^2 = \beta_Q E^2/c$. The spatial components E^i and B^i are the same as the usual expressions for the components of $\mathbf{E}$ and $\mathbf{B}$ for an uniformly moving charge. Inserting these equations into (11) we again find N_s and N_t as in (15). N_s is $\neq 0$ since B is $\neq 0$.

N_s from (15) can be written in the form similar to Eq. (7) from Ref. 1, when the above explicit form of E and B are used. Thus

$$N_s = N^3\gamma_3 = N^{12}\gamma_3 = (\beta_Q ct K_L^2 + (q/c)\beta_Q y(E^\mu u_\mu)\gamma_3. \qquad (16)$$

In the usual approach, e.g., Ref. 1, it is considered that in the S frame the whole physical torque is the 3D $\mathbf{T}$, i.e., T_z, given by Eq. (7) in Ref. 1. We see that in the 4D spacetime the physical torque, theoretically and *experimentally,* is either the bivector N, (14) or (12), or *two* 1-vectors N_s and N_t given by (15) and (16). Only when the laboratory frame S is the γ_0-frame the spatial components of N_s can be put into the correspondence with the components of the 3D $\mathbf{T}$. However note that in (16) all components are the components of the 4D quantities, 1-vectors x, u, K_L, E and B, while in Eq. (7) in Ref. 1 only the corresponding 3D vectors are involved.

Let us now determine N_s and N_t as CBGQs in S'. Relative to the S' frame the charge Q is at rest $u_Q = c\gamma_0'$, but the "fiducial" observers are moving with velocity $v = \gamma_Q c\gamma_0' - \gamma_Q \beta_Q c\gamma_1'$. Then N_s and N_t in S' can be obtained either directly from (10) or by means of the LT (8) of N_s and N_t from (15). We find N_s and N_t as

$$\begin{aligned}
N_s &= N_s'^\mu \gamma_\mu' = N_s'^3 \gamma_3', \;\; N_s'^3 = \gamma_Q \beta_Q N'^{02}, \\
N_t &= N_t'^\mu \gamma_\mu', \;\; N_t'^0 = -\beta_Q \gamma_Q N'^{01}, \;\; N_t'^{1,2} = \gamma_Q N'^{01,2}, \;\; N_t'^3 = 0, \quad (17)
\end{aligned}$$

where N'^{01}, N'^{02} are given in (12). Now N_s is different from zero not only in S but in the S' frame as well. The same results for N_s and N_t in S' can be obtained using (11) and (4) and writing all AQs as CBGQs in the S' frame.

It can be easily seen that N_s (N_t) from (15) is equal to N_s (N_t) from (17); it is the same 4D CBGQ for observers in S and S'; the principle of relativity is naturally satisfied and there is no paradox.

Inserting N_s and N_t from (15) and (17) into Eq. (7), which connects N with N_s and N_t, we find that the expressions for N in S and S' are the same, as it must be.

If we would take that S' is the γ_0-frame, as in Sec. 4.2 in Ref. 2, then the explicit expressions for N_s and N_t as CBGQs would be different than those given in (15) and (17). For example, in that case N_s is zero but $N_t \neq 0$ both in S' and S. However when these new expressions for N_s and N_t as CBGQs are inserted into (7) they will give the same N as when S is the γ_0-frame.

V. CONCLUSIONS

It is proved in this paper, as in Ref. 2, that the physical torques are the 4D geometric quantities, the bivector N defined in (5) and (9), or the 1-vectors N_s and N_t that are derived from N according to (7). They together contain the same physical information as the bivector N. In the considered case N_s and N_t are defined in (10). Only in the γ_0-frame one can deal with components T_i and $T_{t,i}$ of two 3D torques $\mathbf{T}$ and $\mathbf{T}_t$ respectively. In that frame temporal components of N_s and N_t as CBGQs are zero, $N_s^0 = N_t^0 = 0$, and the spatial components are $T_i = N_s^i$ and $T_{t,i} = N_t^i$. All components T_i and $T_{t,i}$ are equally well physical for "fiducial" observers. Hence, it is not true, as generally accepted, that only the 3D torque $\mathbf{T}$ is a well-defined physical quantity. However, it is shown here, and in Ref. 2, that even in the frame of "fiducial" observers the relativistically correct geometric quantities are not 3D vectors $\mathbf{T}$ and $\mathbf{T}_t$, but the bivector N or 1-vectors N_s and N_t. These 4D geometric quantities correctly transform under the LT, e.g., the whole torque N_s remains unchanged under the passive LT, $N_s = N_s^\mu \gamma_\mu = N_s'^\mu \gamma_\mu'$, where the components transform by means of the LT (8) and the basis 1-vectors γ_μ by the inverse LT. This means that the principle of relativity is satisfied and the paradox with the torque does not appear. In contrast to it the components T_i and $T_{t,i}$ transform according to the AT (1), which differ from the LT for components of 1-vectors, e.g., (8). Furthermore, the objections (i) and (ii) from Sec. II show that the transformations of $\mathbf{T}$ and $\mathbf{T}_t$, as geometric quantities in the 3D space, are not the LT but the relativistically incorrect AT.

The validity of the above relations with 4D geometric quantities can be experimentally checked measuring all six independent components of N, or N_s and N_t taken together, in both relatively moving frames. Only such complete data are physically relevant in the 4D spacetime. Remember that the usual 3D torque $\mathbf{T}$ is connected only with three spatial components of N_s in the frame of "fiducial" observers. These three components are not enough for the

determination of the relativistically correct 4D torques N, or N_s and N_t. This is the real cause of Jackson's paradox.

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